

# Schwarzschild black hole as a grand canonical ensemble

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## Abstract

For long black holes have been considered as endowed with a definite temperature. Yet when the Schwarzschild black hole is treated as a canonical ensemble three problems arise: incompatibility with the Hawking radiation, divergence of the partition function, and a formally negative mean-square fluctuation of the energy. We solve all three problems by considering the Schwarzschild black hole as a grand canonical ensemble, with the Hamiltonian (the ADM mass) and the horizon surface area, separately, as observable parameters. The horizon area simulates the number of particles in statistical mechanics since its spectrum is here assumed to be discrete and equally spaced. We obtain a logarithmic correction to the Bekenstein-Hawking entropy and a Gaussian type distribution for the energy levels.

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Black holes (BHs) are conventionally regarded as thermodynamic systems [1,2]. But there exist problems with the description of a black hole as a canonical ensemble [3]. For example, because a BH has negative specific heat, energy fluctuations calculated in canonical ensemble have formally negative variance. The issue of how to give a consistent thermodynamic description of a BH has received renewed urgency with the understanding that the mass spectrum of a BH may be discrete and highly degenerate.

In the early seventies, Bekenstein pointed out that the horizon area of a non-external BH behaves as a classical adiabatic invariant [5,4]. Using Ehrenfest's principle [6] that any classical adiabatic invariant corresponds to a quantum entity with discrete spectrum, Bekenstein conjectured that the spectrum of the horizon area of a (non-extremal) BH should be quantized. He proposed a uniformly spaced spectrum which has later been considered by many authors (see the list in Kastrup [7]). Today the idea of a discrete eigenvalue spectrum for the horizon area is also supported by the work of Ashtekhar and others [8], and recently a uniformly spaced spectrum (for large quantum numbers) has been established by Bojowald and Kastrup [9] within the framework of loop quantum gravity.

Recently, the Schwarzschild BH with the uniformly spaced area spectrum was treated as a microcanonical ensemble by Scharf [10] (a microcanonical ensemble of a gas of neutral and charged BHs has been considered earlier by Harms and Leblanc [11]) and as a quantum canonical ensemble by Kastrup [7] and by Mäkelä and Repo [12]. According to the “no hair” theorems [5,13], an observer outside a Schwarzschild black hole is able to measure only its mass. Hence, all authors cited above have used the mass (energy) of the BH as the sole variable characterizing the system. The main new idea of the present paper is that a Schwarzschild BH formed by gravitational collapse should rather be considered as a grand canonical ensemble. The additional thermodynamic parameter is the horizon surface area of the BH.

In the present paper we shall assume that the area spectrum  $A_n$  of the Schwarzschild BH is given by:

$$A_n = a_0 n \quad n = 0, 1, 2, \dots \quad (1)$$

where  $a_0$  is a constant of the order of the Planck area. Recently, Kastrup and others [14] have shown that Eq. (1) holds true also when the space-time dimensions is taken to be  $D > 4$ .

The area levels  $A_n$  are expected to be degenerate. Denoting the degeneracy by  $g(n)$  and identifying  $\ln g(n)$  with the BH entropy

$$S = A/(4\mathcal{L}_p^2) \quad (2)$$

( $\mathcal{L}_P$  and  $\mathcal{M}_P$  will here denote the Planck length and mass, respectively), Bekenstein and Mukhanov [15] found that  $a_0 = 4\mathcal{L}_p^2 \ln k$ , or equivalently  $g(n) = k^n$  where  $k = 2, 3, 4, \dots$ . They adopted  $k = 2$  for simplicity; recently Hod [16] has argued in favor of the choice  $k = 3$ .

In the quantum mechanical picture, if an observer at infinity makes an attempt to determine the mass of a black hole, his accuracy is limited by the time-energy uncertainty relation as well as by the systematic decrease of the mass of the BH [17]. Thus for a black hole formed by a gravitational collapse, it is impossible, even conceptually, for an observer to know exactly the mass of the black hole. However, if the black hole is in a static state (eternal black hole), the observer, in principle, would be able to determine its mass and then the microcanonical approach would be appropriate.

In the canonical ensemble approach the observer knows only the average value of the mass of the black hole at a given time. In this ensemble the partition function  $Z$  formally diverges because the degeneracy factor  $g(n)$  rises very fast with  $n$ . Kastrup [7] proposes to resolve this problem by defining  $Z$  by analytic continuation, a procedure which is difficult to understand in physical terms. Mäkelä and Repo [12] avoided the problem by studying, not the partition function of the whole spacetime itself, but instead the partition function of the radiation emitted by the BH. We shall see shortly that if we treat the BH as a grand canonical ensemble,  $Z$  is no longer divergent! Before describing this new approach we shall first point out another problem of the canonical approach.

Treating the BH as a canonical ensemble implies that it is in a thermal equilibrium with a surrounding thermal bath. Thus the BH mass remains constant because the density matrix of the canonical ensemble is constant in time; it commutes with the Hamiltonian (the ADM mass operator). However, the mass of an isolated BH formed by a gravitational collapse decreases in time because of losses to Hawking radiation [17], so a description of it via a thermal ensemble seems inappropriate. We propose to solve both problems by abandoning the canonical approach in favor of the grand canonical one.

If the observer is interested in determining the horizon area of the black hole he is limited, apart from the time-energy uncertainty relation, by some kind of area-phase uncertainty relation [18]:

$$\Delta A \Delta \phi \geq \frac{1}{2} a_0. \quad (3)$$

Here  $\phi$  is the canonical conjugate to the area (number) operator. This is the first clue suggesting to treat the black hole as a grand canonical ensemble: the observer only knows the average value of the horizon area (by Eq. (3) the area operator is a number operator, and is thus analogous to the number of particles in a grand canonical system). We shall discuss now further physical grounds for adopting the grand canonical approach for the description of Schwarzschild BHs.

Classically, the ADM mass and the horizon surface area are related by

$$A = \frac{16\pi G^2}{c^2} M^2. \quad (4)$$

Hence, the horizon area was never before considered as a new parameter. However, the mass (energy) and the horizon area (topology feature) of a BH describe two *different* properties.

Consider an observer who is not aware of the classical relation in Eq. (4), and is interested in the BH properties. It is clear that he would use completely different techniques (and apparatus) to measure the mass and the horizon area of the BH. The mass may be measured asymptotically at infinity whereas the horizon surface area should be measured locally. Thus, these two parameters describe operationally distinguishable features of the BH.

In the quantum mechanical picture the distinction between these two parameters becomes prominent. Because of the Hawking phenomenon, the horizon area decreases in time and hence the area operator and the Hamiltonian (the ADM mass operator) do not commute. Thus, the relation in Eq. (4) does not hold true for operators. In [18] we have found the form of the Hamiltonian and have shown that in the classical limit, where the HR is negligible, Eq. (4) is satisfied also for the operators.

Generally, in grand canonical ensembles the number of particles and the energy of the system are taken to be the two observable parameters since neither is constant and each describes a *different* property of the system (even though they are ultimately related by some formula). Since the ADM mass and the horizon area of a Schwarzschild BH have the same relation, we conclude that a Schwarzschild BH should be considered as a grand canonical ensemble!

Assuming the black hole is described by some density operator  $\rho$ , we shall maximize the following quantity (entropy):

$$Q = -\text{Tr}(\rho \ln \rho) - \mu' \langle \mathbf{A} \rangle - b \langle \mathbf{H} \rangle \quad (5)$$

where  $\mu'$  and  $b$  are the Lagrangian multipliers (the physical meaning of these parameters will be discussed later) and  $\mathbf{H}$  is the Hamiltonian (ADM mass) operator (boldface is used everywhere to denote operators). The trace in Eq. (5) may be taken with respect to the area eigenstates; this makes it easy to take the degeneracy factor  $g(n)$  into account. The extremum for  $Q$  under the conditions that  $\text{Tr}(\rho \mathbf{A}) \equiv \langle \mathbf{A} \rangle$  and  $\text{Tr}(\rho \mathbf{H}) \equiv \langle \mathbf{H} \rangle$  with  $\langle \mathbf{A} \rangle$  and  $\langle \mathbf{H} \rangle$  known is attained by

$$\rho = \frac{1}{Z} \exp(-\mu' \mathbf{A} - b \mathbf{H}), \quad (6)$$

where the partition function  $Z$  is defined by

$$Z = \text{Tr}(\exp(-\mu' \mathbf{A} - b \mathbf{H})). \quad (7)$$

Because of the Hawking radiation, the Hamiltonian operator cannot commute with the horizon area operator, because the last operator is not constant in time. Thus  $\rho$  does

not commute with  $\mathbf{H}$ . Hence, we conclude that by choosing the appropriate Hamiltonian operator [18] for the Schwarzschild BH, the density matrix in Eq. (6) would be compatible with BH evolution in the wake of Hawking radiation.

We have shown in ref. [18] that the Hamiltonian (the ADM mass) can be written as

$$\mathbf{H} = \mathbf{M} + \mathbf{V} \quad (8)$$

where  $\mathbf{M} \equiv \sqrt{c^4 \mathbf{A}/16\pi G^2}$  is the mass operator (according to the classical limit) and  $\mathbf{V}$  is a coupling between the horizon area and its canonical conjugate, the phase of the BH. However, as we have pointed out, the interaction term approaches zero as  $\langle \mathbf{M} \rangle \rightarrow \infty$  according to  $\langle \mathbf{V} \rangle \sim \mathcal{M}_p^2 c^2 / \langle \mathbf{M} \rangle$ . Hence, for BHs not near the Planck scales we may neglect the effect of Hawking radiation and assume that  $\mathbf{H} \approx \mathbf{M}$ . Thus, the partition function given in Eq. (7) can be approximated by

$$Z = \sum_{n=0}^{\infty} k^n \exp(-\mu' a_0 n - b m_0 \sqrt{n}) = \sum_{n=0}^{\infty} \exp(-(\mu' a_0 - \ln k)n - b m_0 \sqrt{n}), \quad (9)$$

where  $m_0 \equiv \sqrt{c^4 a_0/16\pi G^2}$  is of the order of the Planck mass. Let us define the dimensionless coefficients  $\alpha \equiv a_0 \mu' - \ln k$  and  $\chi \equiv b m_0$ . We note that the partition function converges for  $\alpha > 0$ ; we thus assume  $\alpha > 0$  and that  $\alpha$  is crudely of order unity. The probability to find the system in the  $n$ th area state can be written as

$$P_n = \frac{1}{Z} \exp(-\alpha n - \chi \sqrt{n}) \quad (10)$$

where  $\alpha$  and  $\chi$  must be reexpressed in terms of  $\langle \mathbf{A} \rangle$  and  $\langle \mathbf{M} \rangle$ .

The relation between  $\langle \mathbf{A} \rangle$  and  $\langle \mathbf{M} \rangle$  is not exactly  $\langle \mathbf{A} \rangle = 16\pi \langle \mathbf{M} \rangle^2 G^2 / c^4$  since the expectation values are not taken with respect to a pure state with sharp mass. What would it be for a BH formed by gravitational collapse? In order to answer, let us first introduce a new parameter  $n_0$ , the number  $n$  that maximizes  $P_n$ . It is clear that around this value the distribution is symmetric, that is

$$P_{n_0+h} \approx P_{n_0-h} \quad (11)$$

for  $h \ll n_0$ . Comparing Eq. (11) with Eq. (10) we find that

$$\chi = -2\sqrt{n_0}\alpha. \quad (12)$$

Eq. (12) raises a serious problem regarding the physical meaning of the parameters  $\mu'$  and  $b$ . A comparison of the partition function (9) with the partition function

$$Z = \sum_n g(E_n) \exp(\beta(\mu N_n - E_n)) \quad (13)$$

written for a general grand canonical ensemble with energy levels  $E_n$ , particle number  $N_n$ , degeneracy  $g(E_n)$ , chemical potential  $\mu$  and inverse temperature  $\beta$  reveals that the parameter  $\mu'$  in our analysis represents the negative of the chemical potential divided by the temperature, and parameter  $b$  the inverse temperature of the BH. Now, Eq. (12) implies that  $\chi$  is negative because  $\alpha > 0$  and thus also  $b = \chi/m_0$  is negative. Does this imply that either the BH's mass or its temperature are negative?

In the view of Mäkelä and Repo [12], one can solve the problem by assuming  $m_0$  is negative. In that case,  $b$  is positive (and thus the temperature too) and  $M_n$  increases when  $n$  decreases. In other words,  $M_n$  becomes greater when the BH becomes smaller. The meaning of  $M_n$  changes: it is not the mass of the BH but, the *energy of the BH radiation* (with an appropriate choice of energy zero). Here we suggest another solution to the problem which also saves the positivity of both the mass and the temperature of the BH, even though  $b$  is negative.

Since the horizon splits the space into two parts, we cannot immediately compare our analysis with the one in general grand canonical ensemble. According to the “no hair” theorems, the degrees of freedom in the interior region are not accessible to an observer at infinity. This affects the independence of the horizon surface area and the mass of the BH. Independence of  $A$  on  $M$  implies, for example, that it is possible to change  $A$  (at least slightly) while keeping  $M$  constant. Hence, at infinity one cannot observe the independence of  $A$  and  $M$  due to the “no hair” theorems.

The partition function that maximizes the entropy of the whole BH spacetime is given by Eq. (9). Now, in order to associate with the BH a temperature  $T_{BH} = \hbar/8\pi M$ , we

must restrict ourselves to the exterior region of the BH; there is no meaning to temperature “inside the BH”. How is this restriction implemented in our scheme?

According to the above arguments we have to make the transition from two independent parameters  $A$  and  $M$  to one parameter. Normally, in the usual grand canonical ensemble, the parameter  $b$  would be given by

$$b = \left( \frac{\partial S}{\partial \langle \mathbf{M} \rangle} \right)_{\langle \mathbf{A} \rangle} . \quad (14)$$

Note that this derivative would be taken with respect to  $\langle \mathbf{M} \rangle$  while keeping  $\langle \mathbf{A} \rangle$  constant, and would be negative. But when we are restricted to the exterior region, there is only one parameter, say  $\langle \mathbf{M} \rangle$ , and therefore the inverse temperature is defined by (the entropy is derived in Eq. (20))

$$\beta \equiv \left( \frac{dS}{d\langle \mathbf{M} \rangle} \right) = \frac{8\pi \langle \mathbf{M} \rangle}{\hbar} + \mathcal{O} \left( \frac{1}{\langle \mathbf{M} \rangle} \right) \quad (15)$$

which is positive and equal to the inverse of Hawking’s temperature. Because the derivative in Eq. (15) is the total derivative, it is distinct from  $b$ , and can be positive. No negative temperature is necessary.

Substituting Eq. (12) back in Eq. (10) we find in the limit  $n_0 \rightarrow \infty$

$$P_n = \frac{1}{Z} \exp(-\alpha(n - 2\sqrt{n_0 n})) \approx \left( \frac{\alpha}{4\pi n_0} \right)^{\frac{1}{2}} \exp \left( -\frac{\alpha}{4n_0} (n - n_0)^2 \right). \quad (16)$$

Hence, we obtain a Gaussian distribution with a variance  $\sigma_A = \sqrt{2n_0/\alpha}$ . Note that  $n_0$  is approximately the average of  $\mathbf{N} \equiv \mathbf{A}/a_0$ . Thus, as is typical of many-particle statistical systems, as  $\langle \mathbf{N} \rangle \rightarrow \infty$ , the absolute fluctuations become large, but the relative fluctuations approach zero.

The entropy may now be expressed as

$$S = -\text{Tr}(\rho \ln \rho) = \mu' \langle \mathbf{A} \rangle + b \langle \mathbf{M} \rangle + \ln Z \quad (17)$$

where for large  $n_0$  the partition function 9 is given up to a very good approximation by

$$Z = \exp(\alpha n_0) \left( \sqrt{\frac{4\pi n_0}{\alpha}} + \mathcal{O}(1/n_0^{1/2}) \right). \quad (18)$$



Calculating  $\langle \mathbf{A} \rangle$ ,  $\langle \mathbf{M} \rangle$  with (16) and taking into account the first order corrections we find

$$\begin{aligned}\langle \mathbf{A} \rangle &= a_0 n_0 + \frac{3a_0}{2\alpha} + O(1/n_0) \\ \langle \mathbf{M} \rangle &= m_0 \sqrt{n_0} + \frac{m_0}{2\alpha \sqrt{n_0}} + O(1/n_0^{3/2}).\end{aligned}\tag{19}$$

Substituting all these in Eq. (17) we finally obtain

$$S = \frac{1}{4\mathcal{L}_p^2} \langle \mathbf{A} \rangle + \frac{1}{2} \ln \left( \frac{\langle \mathbf{A} \rangle}{\mathcal{L}_p^2} \right) + \frac{1}{2} \ln \left( \frac{4\pi}{\alpha} \right) + \frac{3(\ln k - 1)}{2\alpha} + \frac{1}{2}.\tag{20}$$

Note that the main contribution to the entropy is given by  $\langle \mathbf{A} \rangle / 4\mathcal{L}_p^2$  as was to be expected. The logarithmic correction to the entropy is exactly the same as Mäkelä and Repo obtained for the emitted radiation [12] and as Kastrup obtained from his analytic continuation approach [7]. However, our grand canonical approach does not suffer from a divergent partition function. Furthermore, using Eq. (19) to determine the fluctuation in the mass we find

$$\sigma_M^2 \equiv \langle \mathbf{M}^2 \rangle - \langle \mathbf{M} \rangle^2 = \frac{\hbar \ln k}{8\pi\alpha}.\tag{21}$$

Thus, the mean square fluctuations of the energy is *positive* even while the specific heat is *negative*.

In summary the grand canonical approach solves three problems which arise when using the canonical approach with the area spectrum given in Eq. (1). Firstly, the partition function is automatically convergent. Secondly, the grand canonical approach is compatible with Hawking radiation in the sense that it requires the density operator to vary with time. Thirdly, the mean square fluctuations of the energy comes out positive. The distribution of the area (energy) levels is of the Gaussian type (for  $\langle \mathbf{A} \rangle \gg \mathcal{L}_p^2$ ), with relative fluctuations  $\Delta \mathbf{A} / \langle \mathbf{A} \rangle$  of order of  $\mathcal{L}_p / \sqrt{\langle \mathbf{A} \rangle}$ . Furthermore, as a byproduct of the grand canonical approach we have recovered the same logarithmic correction to the entropy earlier derived by Kastrup [7] and by Mäkelä and Repo [12]. Other authors [19–21] have also obtained a logarithmic corrections to the entropy of BHs.

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